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A no-go theorem for off-shell extended supergravities

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Abstract. We prove that there is no possible set of field redefinitions which give linearised off-shell versions of $N \geq 3$ extended supergravity, assuming that the associated global supersymmetry algebras have no central charges and that fermionic auxiliary fields are always monogamous. Our results also prove the absence of off-shell versions of eleven-dimensional simple supergravity.

An important unsolved problem in extended supergravity is to obtain an off-shell formulation of the theory, especially for the maximally extended $N = 8$ version. This is particularly crucial in order to discover if the 'hidden symmetries' discovered recently (Cremmer and Julia 1979) survive sufficiently off-shell in order that the conjectured 63 gauge vector bosons of the hidden local SU(8) symmetry become composite particles or not. Another reason for an off-shell formulation is that the highly efficient supergraph techniques (Grisaru 1982) can then be used to assess the ultraviolet divergence properties of the quantised extended supergravity, which already have indications (Duff 1982) of being considerably alleviated (for $N > 4$) in comparison with other theories of quantum gravity. Simultaneously an off-shell theory will allow a resolution of the structure of the ghost contributions which may (de Wit and van Holten 1978) involve arbitrary high-order interactions in the ghost fields for $N \geq 3$ (when the algebra ceases to close on the new spin-$\frac{1}{2}$ fields).

We will analyse the problem of constructing an off-shell formulation for extended supergravities, following the field redefinition programme used implicitly in one approach to the construction of the $N = 2$ minimal auxiliary fields (de Wit and van Holten 1979), and more extensively in extending that analysis to higher N (Rivelles and Taylor 1981a) as well as in the construction of a linearised superfield formulation of $N = 2$ supergravity (Rivelles and Taylor 1982).

In this method field redefinitions are applied to irreducible representations (irreps) of the global supersymmetry algebra (which we denote by $\mathscr{S}_N$). These irreps are so chosen that there exist field redefinitions which transform the total quadratic (linearised) Lagrangian of the irreps into the linearised off-shell Lagrangian of N-extended supergravity.

The non-existence of any choice of field redefinitions applied to any of a specified set of irreps leading to linearised N-extended supergravity has been established (Rivelles and Taylor 1981a) for $N = 3$ and 4 when the irreps were chosen from a vector, spinor or scalar superfield. Whilst this result indicated that severe difficulties are present in establishing off-shell extended supergravities, it was not complete. We will extend it here to include all irreps of $\mathscr{S}_N$ that could arise in a differential super-geometric

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formulation of extended supergravity based on the existence of a super-bein $E_A{}^M$ and super-connection $\Omega_{AB}{}^C$. Furthermore, we will only consider $N=3$, since we will argue later that off-shell no-go theorems for higher N or higher dimensions follow from that case alone.

There are two questions which must be answered before we can proceed in detail. The first is the nature of the field redefinition rules which we will use; the second is the set of irreps of $\mathcal{S}_3$ which we will consider. Our answer to the first question is that we will use all those available to us. These have been catalogued both for fermions and bosons (Rivelles and Taylor 1982), though as previously (Rivelles and Taylor 1981a) we will not use the boson rules here, since the fermion rules alone will be found to eliminate not only the unwanted fermions (of higher spin or I-spin) but also those required as propagating modes in $N=3$ supergravity.

If we denote by $\pm j$ the quadratic Lagrangian for a fermion of spin j with $\pm$ ve sign for its kinetic energy, the most elementary fermion 'annihilation rule' (so-called because it removes propagating modes, making them auxiliary) is

$$\tfrac{1}{2}-\tfrac{1}{2}\approx 0, \tag{1}$$

the $\approx$ sign denoting that the LHS combines to vanish on-shell. This rule follows easily from the identity

$$\bar{\phi}\not{p}\phi-\bar{\psi}\not{p}\psi=\bar{\lambda}_1\lambda_2$$

with $\lambda_1=\phi+\psi$, $\lambda_2=\not{p}(\phi-\psi)$, and all spinors are Majorana. Similarly for higher-spin fermions we have

$$(\tfrac{3}{2}+\tfrac{1}{2}{}^2)-(\tfrac{3}{2}+\tfrac{1}{2}{}^2)\approx 0, \tag{2}$$

$$(\tfrac{5}{2}+\tfrac{3}{2}{}^2+\tfrac{1}{2}{}^2)-(\tfrac{5}{2}+\tfrac{3}{2}{}^2+\tfrac{1}{2}{}^2)\approx 0. \tag{3}$$

We shall use the rules (1), (2) and (3) and their higher analogues in such a way that the lower-spin partners in (2) and (3) give no further contribution, so the general fermion annihilation rule becomes

$$(j+\tfrac{1}{2})-(j+\tfrac{1}{2})\approx 0 \tag{4}$$

for any non-negative integer j. The rules (4) appear to exhaust all known annihilation rules for fermions; one way to circumvent our no-go theorems is to develop new rules which do not require fermions to be monogamous, marrying off against each other.

The set of irreps which we will consider has already been specified by choosing only those contained in the extended superfields $E_A{}^M$ and $\Omega_{AB}{}^C$. From the representation theory of $\mathcal{S}_N$ (Taylor 1982) we conclude that these have superspin Y and I-spin I values (we use I-spin instead of SU(3) irreps of $\mathcal{S}_3$ since decomposition of products of SO(3) irreps is even simpler than for SU(3)): $Y=\tfrac{7}{2}$, $I=0$; $Y=3$, $I\leqslant 1$; $Y=\tfrac{5}{2}$, $I\leqslant 2$; $Y=2$, $I\leqslant 3$; $Y\leqslant\tfrac{3}{2}$, $I\leqslant 4$.

The basic irrep of $\mathcal{S}_3$ with $Y=I=0$ has fermionic content $\tfrac{3}{2}$, $\tfrac{1}{2}{}^{15+3+3+3}$ (where j^m denotes a fermion m-dimensional irrep of SO(3) of spin j). All other irreps are obtained by multiplying this by irreps of the Lorentz and SO(3) groups and reducing suitably. We can thus construct explicitly the irreps with the above limitations. They are given in table 1.

We may write the general linearised Lagrangian constructed from these irreps as

$$L=\sum_{Y,I}(a_{YI}-b_{YI})L_{YI} \tag{5}$$

Table 1. Table of I-spin assignments I and multiplicities for the various spin components j of the irreps (Y, I) of the supersymmetry algebra $\mathscr{S}_3$. Y denotes superspin, and the entry m^n denotes an I-spin of dimension m with multiplicity n.

J \ (Y, I)	(0, 0)	(0, 1)	(0, 2)	(0, 3)	(0, 4)	(1/2, 0)	(1/2, 1)	(1/2, 2)	(1/2, 3)	(1/2, 4)
3/2	1	3	5	7	9	3	135	357	579	7911
1/2	3^35	$1^33^45^47$	$13^45^47^49$	$35^47^49^411$	$57^49^411^413$	1^235	13^45^27	$13^25^47^29$	$35^27^49^211$	$57^29^411^213$

J \ (Y, I)	(1, 0)	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(3/2, 0)	(3/2, 1)	(3/2, 2)	(3/2, 3)	(3/2, 4)
5/2	1	3	5	7	9	3	135	357	579	7911
3/2	13^35	$1^33^55^47$	$13^45^57^49$	$35^47^59^411$	$57^49^511^413$	1^235	13^45^27	$13^25^47^29$	$35^27^49^211$	$57^29^411^213$
1/2	13^35	$1^33^55^47$	$13^45^57^49$	$35^47^59^411$	$57^49^511^413$	3	135	357	579	7911

J \ (Y, I)	(2, 0)	(2, 1)	(2, 2)	(2, 3)	(5/2, 0)	(5/2, 1)	(5/2, 2)	(3, 0)	(3, 1)	(7/2, 0)
9/2	—	—	—	—	—	—	—	1	3	3
7/2	1	3	5	7	3	135	357	13^35	$1^33^55^47$	1^235
5/2	13^35	$1^33^55^47$	$13^45^57^49$	$35^47^59^411$	1^235	13^45^27	$13^25^47^29$	13^35	$1^33^55^47$	3
3/2	13^35	$1^33^55^47$	$13^45^57^49$	$35^47^59^411$	3	135	357	1	3	—
1/2	1	3	5	7	—	—	—	—	—	—

where a_{YI}, b_{YI} are non-negative integers denoting the multiplicity and sign of the kinetic energy term of the irrep (Y, I) in L, and L_{YI} is the corresponding kinetic energy written in terms of constrained component fields. Thus a fermion of spin $(n+\frac{1}{2})$ will give contribution $\bar{\psi}_n \not{p} \psi_n$, where ψ_n is a symmetric tensor-spinor of rank n, totally symmetric on its vector indices, traceless in any pair and with divergence and γ-trace on any index zero. Any other field representation of spin $(n+\frac{1}{2})$ can always be written in this form by field redefinition. Since we then use all possible field redefinitions to achieve our goal of obtaining an off-shell formulation, such a component field field choice is not restrictive.

We now rewrite (5) as a sum of fields of a given spin. Since we are interested specifically in the fermion contribution we write

$$L_{\mathrm{f}} = \sum_{j,I} d_{j,I} \bar{\psi}_{j,I} \not{p} \psi_{j,I} \tag{6}$$

where ψ_{jI} is the fermion field of spin j and I-spin I, and d_{jI} is a linear combination of the $c_{YI} = (a_{YI} - b_{YI})$. In order that L_{f} correctly describes linearised off-shell $N=3$ supergravity, we require

$$d_{j,I} = 0 \qquad \text{for } j > \tfrac{3}{2}, \quad \text{or } j = \tfrac{3}{2}, I \neq 1, \quad \text{or } j = \tfrac{1}{2}, I > 1 \tag{7}$$

and

$$d_{3/2,1} = d_{1/2,0} = 1, \qquad d_{1/2,1} = -1. \tag{8}$$

The value of $d_{1/2,1}$ is obtained from the 'creation rule' (Rivelles and Taylor 1981b) $(\frac{3}{2} - \frac{1}{2}) = L_{\mathrm{RS}}$, where L_{RS} is the Rarita–Schwinger Lagrangian for a purely massless spin-$\frac{3}{2}$ field. We write the values of $d_{j,I}$ in terms of $C_{Y,I}$ from table 1 as

$$d_{9/2,1} = c_{7/2,0} + c_{3,1}, \qquad d_{9/2,0} = c_{3,0}, \qquad d_{7/2,3} = c_{2,3} + c_{5/2,2} + c_{3,1},$$

$$d_{7/2,2} = c_{7/2,0} + c_{3,0} + c_{2,3} + c_{5/2,1} + c_{5/2,2} + 4c_{3,1},$$

$$d_{7/2,1} = c_{7/2,0} + 3c_{3,0} + c_{5/2,1} + c_{5/2,2} + 5c_{3,1} + c_{2,1} + c_{5/2,0},$$

$$d_{7/2,0} = 2c_{7/2,0} + c_{3,0} + c_{5/2,1} + 3c_{3,1} + c_{2,0}, \qquad d_{5/2,5} = c_{2,3} + c_{3/2,4},$$

$$d_{5/2,4} = 4c_{2,3} + c_{2,2} + c_{3/2,4} + c_{1,4} + c_{5/2,2} + c_{3/2,3},$$

$$d_{5/2,3} = 5c_{2,3} + 4c_{2,2} + c_{3/2,4} + c_{5/2,1} + 2c_{5/2,2} + c_{3,1} + c_{3/2,3} + c_{3/2,2} + c_{1,3} + c_{2,1},$$

$$d_{5/2,2} = c_{3,0} + 4c_{2,3} + 5c_{2,2} + 2c_{5/2,1} + 4c_{5/2,2} + 4c_{3,1} + c_{3/2,3} + c_{5/2,0} + c_{3/2,2} + c_{3/2,1} + 4c_{2,1} + c_{2,0} + c_{1,2},$$

$$d_{5/2,1} = c_{7/2,0} + 3c_{3,0} + c_{2,3} + 4c_{2,2} + 4c_{5/2,1} + 2c_{5/2,2} + 5c_{3,1} + 5c_{2,1} + 3c_{2,0} + c_{1,1} + c_{5/2,0} + c_{3/2,2} + c_{3/2,1} + c_{3/2,0},$$

$$d_{5/2,0} = c_{3,0} + c_{2,2} + c_{5/2,1} + c_{5/2,2} + 3c_{3,1} + 3c_{2,1} + c_{2,0} + c_{1,0} + 2c_{5/2,0} + c_{3/2,1},$$

$$d_{3/2,6} = c_{3/2,4} + c_{1,4}, \qquad d_{3/2,6} = c_{2,3} + 2c_{3/2,4} + 4c_{1,4} + c_{1/2,4} + c_{3/2,3} + c_{1,3},$$

$$d_{3/2,4} = 4c_{2,3} + c_{2,2} + 4c_{3/2,4} + 5c_{1,4} + c_{1/2,4} + c_{0,4} + c_{1/2,3} + 2c_{3/2,3} + c_{3/2,2} + 4c_{1,3} + c_{1,2},$$

$$d_{3/2,3} = 5c_{2,3} + 4c_{2,2} + 2c_{3/2,4} + 4c_{1,4} + c_{1/2,4} + c_{5/2,2} + 4c_{3/2,3} + c_{0,3} + 5c_{1,3} + c_{2,1} + 4c_{1,2} + c_{1,1} + 2c_{3/2,2} + c_{3/2,1} + c_{1/2,2} + c_{1/2,3},$$

$$d_{3/2,2} = 4c_{2,3} + 5c_{2,3} + c_{3/2,4} + c_{1,4} + c_{5/2,1} + c_{5/2,2} + 2c_{3/2,3} + 4c_{1,3} + 4c_{2,1} + c_{2,0} + 5c_{1,2} + 4c_{1,1} + c_{1,0} + c_{0,2} + 4c_{3/2,2} + 2c_{3/2,1} + c_{3/2,0} + c_{1/2,2} + c_{1/2,1} + c_{1/2,3},$$

$$d_{3/2,0}=c_{3,0}+c_{2,2}+c_{5/2,1}+c_{3/2,2}+c_{3/2,1}+2c_{3/2,0}+c_{1/2,1}+c_{2,0}+c_{1,2}+3c_{1,1}+c_{1,0}+c_{0,0}+3c_{2,1},$$

$$d_{1/2,6}=c_{1,4}+c_{1/2,4}+c_{0,4}, \qquad d_{1/2,5}=c_{3/2,4}+4c_{1,4}+2c_{1/2,4}+4c_{0,4}+c_{1/2,3}+c_{0,3}+c_{1,3},$$

$$d_{1/2,4}=c_{3/2,4}+5c_{1,4}+4c_{1/2,4}+4c_{0,4}+2c_{1/2,3}+c_{3/2,3}+4c_{0,3}+c_{1/2,2}+4c_{1,3}+c_{1,2}+c_{0,2},$$

$$d_{1/2,3}=c_{2,3}+c_{3/2,4}+4c_{1,4}+2c_{1/2,4}+4c_{0,4}+4c_{1/2,3}+c_{3/2,3}+4c_{0,3}+5c_{1,3}+4c_{1,2}+c_{1,1}+4c_{0,2}+c_{0,1}+c_{3/2,2}+2c_{1/2,2}+c_{1/2,1},$$

$$d_{1/2,2}=c_{2,2}+c_{1,4}+c_{1/2,4}+c_{0,4}+2c_{1/2,3}+c_{3/2,3}+4c_{0,3}+4c_{1,3}+c_{2,2}+5c_{1,2}+4c_{1,1}+c_{1,0}+4c_{0,2}+4c_{0,1}+c_{0,0}+c_{3/2,2}+c_{3/2,1}+4c_{1/2,2}+2c_{1/2,1}+c_{1/2,0},$$

$$d_{1/2,1}=c_{1/2,3}+c_{0,3}+c_{3/2,2}+c_{3/2,1}+c_{3/2,0}+2c_{1/2,2}+4c_{1/2,1}+c_{1/2,0}+c_{1,3}+c_{2,1}+4c_{1,2}+3c_{1,0}+4c_{0,2}+4c_{0,1}+3c_{0,0}+5c_{1,0},$$

$$d_{3/2,1}=c_{2,3}+4c_{2,2}+c_{5/2,1}+c_{5/2,2}+c_{3,1}+c_{3/2,3}+c_{1,3}+5c_{2,1}+3c_{2,0}+4c_{1,2}+5c_{1,1}+3c_{1,0}+c_{0,1}+c_{5/2,0}+2c_{3/2,2}+4c_{3/2,1}+c_{3/2,0}+c_{1/2,2}+c_{1/2,1}+c_{1/2,0},$$

$$d_{1/2,0}=c_{3/2,1}+c_{1/2,2}+c_{1/2,1}+2c_{1/2,0}+c_{2,0}+c_{1,2}+3c_{1,1}+c_{1,0}+c_{0,2}+3c_{0,1}. \tag{9}$$

The solution of the 23 equations (7) for the $c_{Y,I}$ from (9) gives, after some algebra,

$$d_{1/2,0}=2[c_{1/2,0}-5c_{3,0}+2c_{5/2,1}]. \tag{10}$$

Since this is always an even number, equation (8) can never be satisfied. In other words, the requirements (7) of vanishing on-shell of the unwanted higher spin and I-spin field contributions of the various chosen irreps of $\mathscr{S}_3$ in the linearised Lagrangian (5) also destroys the possibility that the desired physical spin-$\frac{1}{2}$ and -$\frac{3}{2}$ fields for $N=3$ supergravity propagate in the requisite fashion. This was to be expected by a cursory glance at the table, since it does indeed appear difficult to remove the unwanted modes by the annihilation rules (4); the analysis through (7), (8) and (9) confirms this analytically by the result (10).

There are various ways around our no-go theorem for off-shell extension for $N=3$ supergravity.

(i) Super-differential geometry is too restrictive and a larger class of irreps of $\mathscr{S}_3$ must be chosen. This destroys the elegance of the geometric approach, which since the time of Einstein has been important as a general framework within which to construct theories of gravity and matter. More importantly, such an extension would not seem to help us, since irreps of $\mathscr{S}_3$ with higher Y and I values than in table 1 bring along many further annihilation conditions like (7). A cursory examination indicates that what is needed is an infinite set of auxiliary irreps with ever increasing Y and I values. This is clearly a very unsatisfactory solution.

(ii) The annihilation rules (4) can be modified so that fermions no longer have to be annihilated only in pairs but an odd number can be removed by suitable field redefinition rules. Such alternate rules are unknown to the author, so cannot be discussed further.

(iii) Central charges are present in the algebra $\mathscr{S}_3$, so that the irreps no longer have SO(3) (and SU(3)) as symmetry groups for their classification. This possibility has already been suggested elsewhere (Rivelles and Taylor 1981a), and has been used in an attempt to formulate $N=4$ supersymmetric Yang–Mills off-shell (Taylor 1981a, Sohnius *et al* 1980a, b). There are difficulties in such an approach in the non-Abelian case, but it may be the only way out. Indeed, a unique set of multiplets has been discovered recently (Taylor 1982) to allow an off-shell formulation for $N=8$ supergravity in the presence of central charges. More extensive classes of putative auxiliary fields have also been presented (Taylor 1981b) for $N=3$, 4, 5 and 6-extended supergravity in the same central charge situation. In spite of the difficulties arising in the Yang–Mills case, it could well be that the presence of central charges is the answer to our no-go theorem.

We now extend our no-go theorem to N greater than 3. If there were an off-shell formulation of $N>3$ supergravity without central charges, its linearised version could trivially be truncated to $N=3$ to give a contradiction to our result. A similar situation would arise if an off-shell version of 11-dimensional supergravity were obtained, for again it could be reduced to four dimensions by assuming triviality in the other dimensions and then truncated to $N=3$ at the linearised off-shell level.

Our result indicates the need for the construction of extended supergravities in the presence of central charges; the corresponding destruction of the on-shell SO(8) symmetry on going off-shell augurs badly for the binding of the 'hidden symmetry' SU(8) gauge vector bosons required in recent phenomenological applications (Ellis *et al* 1979, Derendinger *et al* 1981) of $N=8$ supergravity.

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